

# Spectral properties of complex networks

Ginestra Bianconi

*The Abdus Salam ICTP, Strada Costiera 11, 34014 Trieste, Italy*

## Abstract

We derive the spectral properties of adjacency matrix of complex networks and of their Laplacian by the replica method combined with a dynamical population algorithm. By assuming the order parameter to be a product of Gaussian distributions, the present theory provides a solution for the non linear integral equations for the spectra density in random matrix theory of the spectra of sparse random matrices making a step forward with respect to the effective medium approximation (EMA) and the single defect approximation (SDA). We extend these results also to weighted networks with weight-degree correlations

PACS numbers:

The interest on the spectral properties of complex networks is growing for the study of their dynamics [1, 2] being relevant for example to understand the stability of ecological networks or the synchronization stability conditions. [3, 4] This problem is related with the investigation of spectral properties of random matrices in random matrices ensembles in many fields of theoretical physics. [5, 6] Starting from the discovery of the Wigner semicircle law in nuclear spectra [7] the Random Matrix Theory has have a wide range of applications from quantum chaos to irreversible classical dynamics and low density liquids [5, 8, 9, 10]

The research on spectral properties of sparse random matrices has started as early as in the 1988 [11] but only recently their relevance has been fully acknowledged for the study of the properties of a number of dynamical models defined on the network, like in the models of synchronization. A number of methods for determining the density of states of random matrices have been proposed which range from the classical results of [12] to the replica method formulation and the supersymmetric formulation. The works of Monasson and Biroli [13, 14, 15] deals with the spectra of Laplacian matrices of random Poissonian networks and small world networks. In [13] a new approximation, the so called single defect approximation (SDA) for the study of the random spectra has been introduced. The approximation has been further improved by the work of Semerjian and Cugliandolo [16] for random Poissonian matrices.

Dorogovtsev et al. [1, 17, 18], have developed random walk based methods for the evaluation of the spectra of the adjacency matrix and the spectra of the Laplacian of complex networks.

Using the replica method as in [13, 15, 16] the problem reduces to making a good replica symmetric ansatz for the functional order parameter defined on a vector of continuous variables defined in the real axis. In the contest of a statistical mechanics model for studying the fluxes in the metabolic network [19, 20] a similar technical problem was solved assuming that the functional order parameter can be written as a weighted sum of Gaussians. The problem was then solved by proposing a population dynamics to find the statistical weights corresponding to each Gaussian in the sum [20]. In this work, following reference [19, 20] we will derive the spectra of random matrices using the replica method and the development of the order parameter in term of weighted Gaussians, a technique that allows also for the extension to weighted matrices of random entries. This method can be applied both to adjacency matrices and to Laplacian matrices providing the tools for the calculation of

different properties of the graph.

## I. SPECTRA OF A MATRIX

Given a random matrix of eigenvalues  $\lambda_n$  with  $n = 1 \dots N$ , the spectral density  $\rho(\lambda)$  is defined as

$$\rho(\lambda) = \frac{1}{N} \sum_n \delta(\lambda - \lambda_n) \quad (1)$$

and can also be expressed as

$$\rho(\lambda) = -\frac{1}{\pi N} \text{ImTr} \frac{1}{\lambda + i\epsilon - A} \quad (2)$$

We suppose that in the thermodynamics limit the spectral density is self-averaging, i.e.

$$\rho(\lambda) \rightarrow \langle \rho(\lambda) \rangle \quad (3)$$

where the average is performed over all matrices in a given ensemble. To solve the spectra of the a matrix  $A$  in a given ensemble of random matrix we introduce the generating function  $\Gamma(\lambda)$

$$\Gamma(\lambda) = \frac{1}{Z_\phi} \int \prod_{i=1}^N \prod_{a=1}^n d\phi_i^a \prod_{i,a} \exp\left(\frac{i}{2} \lambda \phi_i^a \phi_i^a\right) \prod_{\langle i,j \rangle} \left\langle e^{-i \sum_a \phi_i^a A_{ij} \phi_j^a} \right\rangle \quad (4)$$

with

$$Z_\phi = \int \prod_{i,a} d\phi_i^a e^{\sum_{i,a} \phi_i^a \phi_i^a}. \quad (5)$$

The spectral density is given by

$$\rho(\lambda) = \lim_{n \rightarrow 0} \frac{-2}{\pi n N} \text{Im} \frac{\partial}{\partial \lambda} \langle \Gamma(\lambda) \rangle \quad (6)$$

## II. SPECTRA OF ADJACENCY MATRIX OF SPARSE NETWORKS

To solve the spectra of the adjacency matrix of a random complex networks we introduce the generating function  $\Gamma(\lambda)$

$$\Gamma(\lambda) = \frac{1}{Z_\phi} \int \prod_{i=1}^N \prod_{a=1}^n d\phi_i^a \prod_{i,a} \exp\left(\frac{i}{2} \lambda \phi_i^a \phi_i^a\right) \prod_{\langle i,j \rangle} \left\langle e^{-i \sum_a \phi_i^a a_{ij} \phi_j^a} \right\rangle. \quad (7)$$

We assume that the support of our matrix is a random uncorrelated network with given expected degree assigned to each node of the network i.e. a realization of the random hidden-variable model [24, 25, 26, 27, 28]. In particular we fix the expected degree distribution of

each node  $i$  of the undirected network to be  $q_i$  and we assume that the matrix elements  $a_{i,j}$  are distributed following

$$P(a_{i,j}) = \frac{q_i q_j}{\langle q \rangle N} \delta(a_{i,j} - 1) + \left(1 - \frac{q_i q_j}{\langle q \rangle N}\right) \delta(a_{i,j}), \quad (8)$$

for  $i < j$  ( $a_{ij} = a_{ji}$ ) and where  $\delta()$  indicates the Kronecker delta. The partition function can then be average over the network ensembles

$$\begin{aligned} \langle \Gamma(\lambda) \rangle &= \frac{1}{Z_\phi} \int \prod_{i=1}^N \prod_{a=1}^n d\phi_i^a \prod_{i,a} \exp\left(\frac{i}{2} \lambda \phi_i^a \phi_i^a\right) \\ &\times \exp \left\{ -\frac{1}{2} \sum_{i,j} \frac{q_i q_j}{\langle q \rangle N} [1 - \exp(i \sum_a \phi_i^a \phi_j^a)] + \mathcal{O}(N^0) \right\}. \end{aligned} \quad (9)$$

We introduce the order parameters of the replicated variables on sparse networks [14]

$$c_q(\vec{\phi}) = \frac{1}{N_q} \sum_i \delta(q_i - q) \prod_a \delta(\phi_i^a - \phi^a) \quad (10)$$

getting for the partition function an expression of the type

$$\langle \Gamma(\lambda) \rangle = \int \mathcal{D}c_q(\vec{\phi}) \exp[nN\Sigma(\{c_q(\vec{\phi})\})]$$

with

$$\begin{aligned} n\Sigma &= - \sum_q \int d\vec{\phi} p_q c_q(\vec{\phi}) \ln(c_q(\vec{\phi})) + i \sum_q p_q c_q(\vec{\phi}) \frac{1}{2} \lambda \sum_a \phi^a \phi^a \\ &- \int d\vec{\phi} \int d\vec{\psi} \sum_{qq'} p_q p_{q'} \frac{1}{2} \frac{qq'}{\langle q \rangle} c_q(\vec{\phi}) c_{q'}(\vec{\psi}) (1 - \exp(i\vec{\phi} \cdot \vec{\psi})) + \mathcal{O}(N^{-1}) \end{aligned} \quad (11)$$

$$(12)$$

The saddle point equations for evaluating  $\Sigma$  are given by

$$\begin{aligned} c_q(\vec{\phi}) &= \exp \left\{ i \frac{\lambda}{2} \sum_a \phi^a \phi^a - q[1 - \hat{c}(\vec{\phi})] \right\} \\ \hat{c}(\vec{\phi}) &= \sum_{q'} \frac{q' p_{q'}}{\langle q \rangle} \int d\vec{\psi} c_{q'}(\vec{\psi}) \exp(i\vec{\phi} \cdot \vec{\psi}). \end{aligned} \quad (13)$$

We assume that the solution of the saddle point equation is replica symmetric, i.e. the distribution of the variables  $\phi_a$  conditioned to a vector field  $\vec{x}$  are identically equal distributed,

$$c(\vec{\phi}) = \int d\vec{x} P(\vec{x}) \prod_{a=1}^n \Psi(\phi_a | \vec{x}) \quad (14)$$

where  $\Psi(\phi|\vec{x})$  are distribution functions of  $\phi$  and  $P(\vec{x})$  is a probability distribution of the vector field  $\vec{x}$ . For the function  $\Psi(\phi|\vec{x})$  the exponential form is usually assumed in Ising models. In our continuous variable case for our quadratic problem, we assume instead, as in [20], that  $\Psi(\phi|\vec{x})$  has a Gaussian form. This assumption could be in general considered as an approximate solution of the equations (13). Explicitly we assume that the functions  $c_q(\vec{\psi})$ ,  $\hat{c}(\vec{\psi})$

$$\begin{aligned} c_q(\vec{\phi}) &= \int dh_q P_q(h_q) \prod_a \exp \left[ -\frac{1}{2} h_q \phi^a \phi^a \right] \left( \sqrt{\frac{h_q}{2\pi}} \right)^n \\ \hat{c}(\vec{\phi}) &= \int d\hat{h} \hat{P}(\hat{h}) \prod_a \exp \left[ -\frac{1}{2} \hat{h} \phi^a \phi^a \right]. \end{aligned} \quad (15)$$

The saddle point equations (13), taking into account the expression for the order parameters (15) closes as in the problem studied in [20] and can be written as recursive equation for  $P_q(h_q)$ ,  $\hat{P}(\hat{h})$ , i.e.

$$\begin{aligned} P_q(h_q) &= \sum_k e^{-q} q^k \frac{1}{k!} \int \dots \int \prod_{l=1}^k d\hat{h}^l \hat{P}(\hat{h}^l, \hat{m}^l) \delta \left( h_q - \sum_{l=1}^k \hat{h}^l - i\lambda \right) \\ \hat{P}(\hat{h}) &= \sum_q \frac{qp_q}{\langle q \rangle} \int dh_q \prod_i P_q(h_q) \delta \left( \hat{h} - \frac{1}{h_q} \right). \end{aligned} \quad (16)$$

Once the distributions  $P_q(h_q)$  are found by the population dynamics algorithm, then the spectral density of the network can be expressed as

$$\rho(\lambda) = -\frac{1}{\pi} \sum_q p_q \int dh_q P(h_q) \text{Im} \frac{i}{h_q} \quad (17)$$

Equations 16 can be solved as suggested in [20] by a population dynamics algorithm. The action of the algorithm for finding  $\hat{P}(\hat{h})$  is summarized in the following pseudocode

**algorithm** PopDyn( $\{\hat{h}\}$ ) **begin do**

- select a random index  $\alpha \in (1, M)$
- choose a random  $q$  with probability  $qp_q$
- draw  $k$  from a Poisson distribution ( $e^{-q_i} q_i^k / k!$ )
- select  $k$  indexes  $\beta_1, \dots, \beta_k \in \{1, \dots, M\}$

$$\hat{h}^\alpha : = \frac{1}{i\lambda + \sum_{l=1}^k h_{\lambda}^{\beta_l}}; \quad (18)$$

**while** (not converged) **return end**

The effective medium approximation (*EMA*) as found in [17] will be given by the solution of the population dynamics with  $M = 1$ , i.e.

$$\hat{h}_{EMA} = \sum_q \frac{qp_q}{\langle q \rangle} \frac{1}{i\lambda + q\hat{h}_{EMA}}. \quad (19)$$

The density in this approximation take the form

$$\rho(\lambda) = \sum_q p_q \text{Im} \frac{1}{\lambda - iq\hat{h}_{EMA}} \quad (20)$$

### III. SPECTRA OF THE LAPLACIAN

The Laplacian of a complex networks plays a crucial role in diffusion process on the network and on the stability of many dynamical fixed points [3, 4, 18]. The Laplacian is defined in terms of the adjacency matrix  $a_{ij}$  of the network as the matrix of entries  $L_{ij} = -a_{i,j} + \sum_k a_{ik}\delta_{ij}$ . For the Laplacian matrix the generating function  $\Gamma(\lambda)$  takes the form,

$$\Gamma(\lambda) = \frac{1}{Z_\phi} \int \prod_{i=1}^N \prod_{a=1}^n d\phi_i^a \prod_{i,a} \exp\left(\frac{i}{2}\lambda\phi_i^a\phi_i^a\right) \prod_{\langle i,j \rangle} \left\langle e^{-i\sum_a \phi_i^a L_{ij} \phi_j^a} \right\rangle. \quad (21)$$

Performing the average over the networks in hidden variable ensemble with fixed expected degree, Eq. (8), we obtain

$$\begin{aligned} \langle \Gamma(\lambda) \rangle &= \frac{1}{Z_\phi} \int \prod_{i=1}^N d\phi_i^a \prod_{i,a} \exp\left(i\frac{1}{2}\lambda\phi_i^a\phi_i^a\right) \\ &= \exp \left\{ -\frac{1}{2} \sum_{i,j} \frac{q_i q_j}{\langle q \rangle N} \left[ 1 - \exp \left( \frac{i}{2} (\phi_i^a - \phi_j^a)^2 \right) \right] + \mathcal{O}(N^0) \right\}. \end{aligned} \quad (22)$$

Introducing the order parameters  $c_q(\vec{\phi})$  defined in Eq. (10) we get for the partition function

$$\langle \Gamma(\lambda) \rangle = \int \mathcal{D}c_q(\vec{\phi}) \exp[nN\Sigma(\{c_q(\vec{\phi})\})]$$

with

$$\begin{aligned} n\Sigma &= - \sum_q \int d\vec{\phi} p_q c_q(\vec{\phi}) \ln(c_q(\vec{\phi})) + \frac{i}{2}\lambda \sum_q p_q c_q(\vec{\phi}) \sum_a \phi^a \phi^a \\ &\quad - \frac{1}{2} \int d\vec{\phi} \int d\vec{\psi} \sum_{qq'} p_q p_{q'} \frac{1}{2} \frac{qq'}{\langle q \rangle} c_q(\vec{\phi}) c_{q'}(\vec{\psi}) \left\{ 1 - \exp \left[ \frac{i}{2} (\vec{\phi} - \vec{\psi})^2 \right] \right\} + \mathcal{O}(N^{-1}). \end{aligned} \quad (23)$$

$$(24)$$

The saddle point equation determining the order parameter are

$$\begin{aligned} c_q(\vec{\phi}) &= \exp \left\{ i \frac{\lambda}{2} \sum_a \phi^a \phi^a - q[1 - \hat{c}(\vec{\phi})] \right\} \\ \hat{c}(\vec{\phi}) &= \sum_{q'} \frac{q' p_{q'}}{\langle q \rangle} \int d\vec{\psi} c_{q'}(\vec{\psi}) \exp \left[ \frac{i}{2} (\vec{\phi} - \vec{\psi})^2 \right]. \end{aligned} \quad (25)$$

Again these equations can be solved with the Gaussian ansatz introduced in [20], Eq. (15),

$$\begin{aligned} P_q(h_q) &= \sum_k e^{-q} q^k \frac{1}{k!} \int \dots \int \prod_{l=1}^k d\hat{h}^l \hat{P}(\hat{h}^l, \hat{m}^l) \delta \left( h_q - \sum_{l=1}^k \hat{h}^l - i\lambda \right) \\ \hat{P}(\hat{h}) &= \sum_q \frac{q P_q}{\langle q \rangle} \int dh_q \prod_i P_q(h_q) \delta \left( \hat{h} - \frac{1}{h_q - i} + i \right) \end{aligned} \quad (26)$$

Finally the spectral density is given by

$$\rho(\lambda) = -\frac{1}{\pi} \sum_q p_q \int dh_q P(h_q) \text{Im} \frac{i}{h_q}. \quad (27)$$

Equations (24) can again be solved by a population dynamics algorithm The action of the algorithm for finding  $\hat{P}(\hat{h})$  is summarized in the following pseudocode

**algorithm** PopDyn( $\{\hat{h}\}$ ) **begin do**

- select a random index  $\alpha \in (1, M)$
- choose a random  $q$  with probability  $qp_q$
- draw  $k$  from a Poisson distribution ( $e^{-q_i} q_i^k / k!$ )
- select  $k$  indexes  $\beta_1, \dots, \beta_k \in \{1, \dots, M\}$

$$\hat{h}^\alpha := \frac{1}{i(\lambda - 1) + \sum_{l=1}^k h_\lambda^{\beta_l}} - i; \quad (28)$$

**while** (not converged) **return end** Once the distribution of  $\hat{h}$  is found from the first equation of (26) it is strait-forward to calculate the distributions for  $P_q(h_q)$ . The effective medium approximation (EMA) as found in will be given by the solution of the population dynamics with  $M = 1$ , i.e.

$$\hat{h}_{EMA} = \sum_q \frac{qp_q}{\langle q \rangle} \frac{1}{i(\lambda - 1) + q\hat{h}_{EMA}} - i. \quad (29)$$

The density in this approximation take the form

$$\rho(\lambda) = \sum_q p_q \text{Im} \frac{1}{\lambda - iq\hat{h}_{EMA}} \quad (30)$$

#### IV. WEIGHTED NETWORKS

The over-mentioned results can be extended to weighted networks with weight degree correlations. The correlation between the weight of the links  $A_{ij}$  ending to a node  $i$  and the degree of the node  $i$  have been observed in different networks [29] and can also be explained by growing network models [30]. A network ensemble with weight degree correlations can be formulated by assuming that the weight of a link between node  $i$  and node  $j$ , if present, has a value  $w_{ij} = C(q_i q_j)^\theta$  where  $q_i$  and  $q_j$  are the expected conductivities of node  $i$  and  $j$  and  $C, \theta$  are two parameters specifying the ensemble under consideration. Therefore in the following we will consider the symmetrix matrix  $w_{ij}$  with distribution of the matrix elements given by

$$P(w_{i,j}) = \frac{q_i q_j}{\langle q \rangle N} \delta(w_{i,j} - C(q_i q_j)^\theta) + \left(1 - \frac{q_i q_j}{\langle q \rangle N}\right) \delta(w_{i,j}). \quad (31)$$

for  $i < j$  and  $w_{ij} = w_{ji}$  The generating function  $\Gamma(\lambda)$  for this ensemble of networks is given by

$$\Gamma(\lambda) = \frac{1}{Z_\phi} \int \prod_{i=1}^N \prod_{a=1}^n d\phi_i^a \prod_{i,a} \exp\left(\frac{i}{2} \lambda \phi_i^a \phi_i^a\right) \prod_{\langle i,j \rangle} \left\langle e^{-i \sum_a \phi_i^a w_{ij} \phi_j^a} \right\rangle. \quad (32)$$

with its average over the distribution (31) taking the usual form

$$\langle \Gamma(\lambda) \rangle = \int \mathcal{D}c_q(\vec{\phi}) \exp[nN \Sigma(\{c_q(\vec{\phi})\})].$$

with  $c_q(\vec{\phi})$  given by (10) and

$$\begin{aligned} n\Sigma = & - \sum_q \int d\vec{\phi} p_q c_q(\vec{\phi}) \ln(c_q(\vec{\phi})) + \frac{i}{2} \lambda \sum_q p_q c_q(\vec{\phi}) \sum_a \phi^a \phi^a \\ & - \frac{1}{2} \int d\vec{\phi} \int d\vec{\psi} \sum_{qq'} p_q p_{q'} \frac{1}{2} \frac{qq'}{\langle q \rangle} c_q(\vec{\phi}) c_{q'}(\vec{\psi}) [1 - \exp(i(C(qq')^\theta \vec{\phi} \cdot \vec{\psi})^2)] + \mathcal{O}(N^{-1}) \end{aligned} \quad (33)$$

The saddle point equation to be solved are



$$\begin{aligned}
c_q(\vec{\phi}) &= \exp \left\{ \frac{1}{2} \lambda \sum_a \phi^a \phi^a - q[1 - \hat{c}_q(\vec{\phi})] \right\} \\
\hat{c}_q(\vec{\phi}) &= \sum_{q'} p_{q'} \frac{q'}{\langle q \rangle} \int c_{q'}(\vec{\psi}) d\vec{\psi} \exp \left[ iC(qq')^\theta \vec{\phi} \cdot \vec{\psi} \right].
\end{aligned} \tag{34}$$

The recursive equations to be solved at the saddle point are

$$\begin{aligned}
P_q(h_q) &= \sum_k e^{-q} q^k \frac{1}{k!} \int \dots \int \prod_{l=1}^k d\hat{h}_q^l \hat{P}_q(\hat{h}_q^l) \delta \left( h_q - \sum_{l=1}^k \hat{h}_q^l - i\lambda \right) \\
\hat{P}_q(\hat{h}_q) &= \sum_{q'} \frac{q' P_{q'}}{\langle q \rangle} \int dh_{q'} P_{q'}(h_{q'}) \delta \left( \hat{h}_q - \frac{C^2(qq')^{2\theta}}{h_{q'}} \right)
\end{aligned} \tag{35}$$

The spectral density is given by

$$\rho(\lambda) = -\frac{1}{\pi} \sum_q p_q \int dh_q P(h_q) \text{Im} \frac{i}{h_q} \tag{36}$$

The equations (35) can be solved by a population-dynamical algorithm.

The action of the algorithm is summarized in the following pseudo code

**algorithm** PopDyn( $\{\hat{h}_q\}$ ) **begin do**

- select a random  $q$  and a random index  $\alpha \in (1, M)$
- choose a random  $q'$  with probability  $q' p_{q'}$
- draw  $k$  from a Poisson distribution ( $e^{-q'} q'^k / k!$ )
- select  $k$  indexes  $\beta_1, \dots, \beta_k \in \{1, \dots, M\}$

$$\hat{h}_q^\alpha := \frac{C^2(qq')^\theta}{i\lambda + \sum_{l=1}^k \hat{h}_{q'}^{\beta_l}}; \tag{37}$$

**while** (not converged) **return end** The equivalent of the effective medium approximation are the following equation for  $\hat{h}_q^{(EMA)}$

$$\hat{h}_q^{(EMA)} = \sum_{q'} \frac{q' p_{q'}}{\langle q \rangle} \frac{C^2(qq')^\theta}{i\lambda + q' \hat{h}_{q'}^{(EMA)}} \tag{38}$$

and the spectral density is given by

$$\rho(\lambda) = \sum_q p_q \text{Im} \frac{1}{\lambda - iq\hat{h}_q^{(EMA)}} \quad (39)$$

In conclusion we have provided a solution for the non linear integral equations for the spectra density in random matrix theory of the spectra of sparse random matrices introducing the order parameter as product of Gaussian distributions, the applications of this approach will be relevant in many fields and stability of stationary state in dynamical system defined on complex networks.

After this work was completed we become aware of similar findings obtained by R. Kuehn [31].

- 
- [1] S. N. Dorogovtsev, A. V. Goltsev, J.F. F. Mendes, arXiv:0705.0010 (2007).
  - [2] R. Albert and A.-L. Barabasi Rev. Mod. Phys. **74**, 47 (2002).
  - [3] R. M. May, Nature, **238**, 413 (1972).
  - [4] T. Nishikawa, A. E. Motter, Y.-C. Lai and F. C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003).
  - [5] T. Guhr, A. Mueller-Groeling, H. A. Weidenmueller Phys. Rept. **299**, 189 (1998).
  - [6] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, S. S. M. Wong Rev. Mod. Phys. **53**, 385 (1981).
  - [7] Eugene P. Wigner "Random Matrices in Physics", SIAM Review, **379**, pp. 1-23 (1967).
  - [8] A. V. Andreev, O. Agam, B. D. Simons, and B. L. Altshuler Phys. Rev. Lett. **76**, 3947-3950 (1996).
  - [9] V. E. Kravtsov and K. A. Muttalib Phys. Rev. Lett. **79**, 1913 (1997).
  - [10] A. Cavagna, I. Giardinà, and G. Parisi, Phys. Rev. Lett. **83**, 108 (1999).
  - [11] G. J. Rodgers and A. J. Bray, Phys. Rev. B **37**, 3557 (1988).
  - [12] M. L. Mehta, *Random Matrices* (Elsevier, Amsterdam, 1983).
  - [13] G. Biroli and R. Monasson, Jour. Phys. A **32**, L255 (1999).
  - [14] R. Monasson, Journ. Phys. A **31**, 513 (1998).
  - [15] R. Monasson, Eur. Phys. Jour. B **12**, 555 (1999).
  - [16] S. Semerjian, L. F. Cugliandolo, Jour. Phys. A **35**, 4837 (2002).

- [17] S. N. Dorogovtsev, A. V. Goltsev, J. F. F. Mendes and A. N. Samukhin, Phys. Rev. E **68**, 046109 (2003).
- [18] A. N. Samukhin, S.N. Dorogovtsev and J. F. F. Mendes, arXiv:0706.1176 (2007).
- [19] G. Bianconi and R. Zecchina in European Conf. on Complex Systems 2007, Dresden October 1-5 2007, (2007).
- [20] G. Bianconi and R. Zecchina, Networks and Heterogeneous Media, in press (2008).
- [21] J. J. M. Verbaaschot, H. A. Weidermüller and M. R. Zirbauer, Phys. Rep. **129**,367 (1985).
- [22] C.Itoi, H. Mukaida, Y. Sakamoto, Jour. Phys. A ,**30**, 5709 (1997).
- [23] Y. V. Fyodorov, J. Phys. A, **32**, 7429 (1996); Y. V. Fyodorov, and A. D. Mirlin, J. Phys. A **24**, 2219 (1991).
- [24] J. Park and M. E. J. Newman, Phys. Rev. E **70**, 066146 (2004).
- [25] B. Sodeberg, Phys. Rev. E **66**, 066121 (2002).
- [26] F. Chung and L. Lu, PNAS **99**, 15879 (2002).
- [27] G. Caldarelli, A. Capocci, P. De Los Rios and M. A. Muñoz, Phys. Rev. Lett. **89**, 258702 (2002).
- [28] M. Boguñá and R. Pastor-Satorras, Phys. Rev. E **68**, 036112 (2003).
- [29] A. Barrat, M. Barthélemy, R. Pastor-Satorras and A. Vespignani, PNAS **101**, 3747 (2004).
- [30] G. Bianconi, Europhys. Lett. **71**, 1029 (2005).
- [31] R. Kuehn, arXiv:0803.2886v2 [cond-mat] (2008).